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COMMON FIXED POINT RESULTS IN GENERALIZED BANACH SPACE

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ABSTRACT

In this paper, we have obtained some common fixed theorem on generalized Banach space which is an extansion of some well known result of [12].

KEYWORDS: Generalized normed linear space, Generalized space, Generalized Banach space.

1. Introduction

Fixed point theory plays a basic role in application of Various Branches of Mathematics,from elementary calculus and linear Alzebra to Toplogy and Analysis. It is not herisncted restected to in mathematics and this theory has many application and in other decipline. This theory is closely related to Game Theory, Melatry, Economics, statastics, and Medicines.

The Fixed point method Specailly Banach Contraction Principle provides a power full tool for obtaining the solution for these equation which where very difficult to solve by any other methods. No dout,it is also true that some qualitative properties of the solution of related equationis lost by Functional Analysis approach many attempt have been made in this direction to formulate Fixed point theorems include contraction as well as Contractive Mapping.

Browder[1]was the first mathematician to study Non Expansive Mapping,He applied this result for proving the Existance of solutions of certain integral equation.

Browder[1], Gohde[6] and Kirk[11] have independently proved a fixed theorem for Non Expensive Mapping defined on a closed bounded and convex subset of a uniformly convex Banach space and the space with richer generalization of non-expansive mappings, prominent being Datson[4]. Emmanuele[5], Goebel[7], Goebel and Zlotkienwicz[8], Isiki[10],Sharma and Rajpur[13],Singh and Chatterjee[14].They have derived valuable result with non-contraction mapping in Banach space.

Recently described about the application of Banach's contraction principle[2],Ghalar[9] introduced the concept of 2-Banach.Resently Badshah and Gupta[3], Yadav,Rajput and Bhardwaj[15] and Yadav,Rajput,Choudhary and Bhardwaj[16] also worked for Banach and 2-Banach space for non contraction mapping.

In this manuscript,the known result [12] is extending generalized Banach space where the extension of common fixed point for generalized Banach space is investigated.

We have proved common fixed point theorem in generalized Banach space.

2. Preliminaries:

we recall some definition and properties of generalized linear space.

Definition 2.1*:* A Norm linear space N is called Banach space if it is complete,that is every Cauchy sequence in N convergent to a point N.

Definition 2.2: If X($\neq \phi$)is a linear space havings(>=)$\in R$ Let$\|.\|$ denotd a function from linear space X into R that satisfies the following axioms:

1) $\forall\, x \in X, \|x\| \geq 0, \|x\| = 0 \; iff \; x = 0$
2) $\forall\, x, y \in X, \|x + y\| \leq s\{\|x\| + \|y\|\}$
3) $\forall\, x \in X, \alpha \in R, \|\alpha x\| = |\alpha|\|x\|$

$\|x\|$ is called norm of $x$and (X,$\|.\|$) is called generalized normed linear space.if for s=1,it reduces to standard normed linear space**.**

Definition 2.3: A linear generalized normed space in which every sequence is convergent is called generalized Banach space**.**

Definition 2.4: The generalizd Banach space is complete if every Cauchy sequence convergences.

Definition 2.5: If X($\neq \phi$) is a linear space havings(>=)$\in R$ Let$\|.,.\|$ denotd a function from linear space X into R that satisfies the following axioms such that for $x, y, z \in X$

1) $\|x, y\|$ =0 $iff\ x$ &y are linearly dependent
2) $\|x, y\| = \|y, x\|$
3) $\|x, \beta y\| \leq |\beta| \|x, y\|, \beta real$
4) $\|x, y + z\| \leq s[\|x, y\| + \|x, z\|]$,

$\|x\|$ is called norm of x and $(M, \|.,.\|)$ is called generalized 2-normed linear space.If for s=1,it reduce to standard 2-normed linear space.

Definition 2.6: A linear generalized 2-normd space in which every sequence is convergent is called generalized 2-Banach space.

To prove our main result we will use the following lemma.

Lemma2.7: Suppose$(X, \|.,.\|)$ be a generalized space and $\{y_{2n}\}$ be a sequence in X such that

$\|y_{2n+1} - y_{2n+2}\| \leq \lambda \|y_{2n} - y_{2n+1}\|, n = 0,1,2 \dots \dots$ (2.7.1)

where $0 \leq \lambda < 1$ then the sequence is Cauchy in X provided s$\lambda < 1$. .

Now in section I, we will find some fixed point theorem in generalized Banach space.

3. Main result :

Theorem3.1: Let X be a Generalized Banach space with $\|.,.\|$ and let $T_1, T_2: X \to X$ be a function with the following mapping:

$\|T_1(x) - T_2(y)\| \leq a\ \|x - T_1 x\| + b\ \|y - T_2(y)\| + c\ \|x - y\|$ (3.1.1)
$\forall x, y \in X$Where a,b and c are non negative real number and satisfy $a + s(b + c) < 1 for s \geq 1$,then T_1 & T_2 have a unique common fixed point.

Proof: Let $x_0 \in X$ and $\{x_{2n}\}\ and\ \{x_{2n+1}\}$ be any two sequence in X such that

$x_{2n} = T_1 x_{2n-1} = {T_1}^{2n} x_0$ (3.1.2)

$x_{2n+1} = T_2 x_{2n} = {T_2}^{2n+1} x_0$ (3.1.3)

Note that, if $x_{2n} = x_{2n+1}$ for some n ≥ 0, then x_{2n} is fixed point of $T_1\ and\ T_2$ .Now putting $x = x_{2n}\ and\ y = x_{2n-1}$.from(3.1.1),wehave

$\|x_{2n+1} - x_{2n}\|$=$\|T_1 x_{2\,n} - T_2 x_{2n-1}\|$

$$\leq a\|x_{2n} - T_1 x_{2n}\| + b\|x_{2n-1} - T_1 x_{2n-1}\| + c\|x_{2n} - x_{2n-1}\|$$

$$= a\|x_{2n} - x_{2n+1}\| + b\|x_{2n-1} - x_{2n}\| + c\|x_{2n} - x_{2n-1}\|$$

$$\Rightarrow \quad (1 - a)\|x_{2n} - x_{2n-1}\| \leq (b + c)\|x_{2n} - x_{2n-1}\|$$

$$\Longrightarrow \|x_{2n} - x_{2n-1}\| \leq \left(\frac{b + c}{1 - a}\right) \|x_{2n} - x_{2n-1}\|$$

$=h\|x_{2n} - x_{2n-1}\|$ where h=$\frac{b+c}{1-a}$

Continuing this process we can easily say that $\|x_{2n} - x_{2n-1}\| \leq h^{2n} \|x_1 - x_0\|$

This implies that $T_1\ and\ T_2$ are contraction mapping.Now it is to show that $\{x_{2n}\}$ is Cauchy sequence in X.

Let m,n> 0 with $m > n$ then from (3.1.1)we have

$$\|x_{2n} - x_{2m}\| \leq s\{\|x_{2n} - x_{2m}\| + \|x_{2n+1} - x_{2m}\|\}$$

$\leq s\|x_{2n} - x_{2n+1}\| + s^2 \left\|x_{2n+1} - x_{2n+|2}\right\| + s^3 \left\|x_{2n+|2} - x_{2n+3}\right\| + \dots\dots\dots\dots\dots\dots\dots.$

$\leq sh^{2n} \|x_0 - x_1\| + s^2 h^{2n+1} \|x_0 - x_1\| + s^3 h^{2n+2} \|x_0 - x_1\| + \dots\dots\dots\dots\dots\dots\dots\dots$

$= sh^{2n}\ \|x_0 - x_1\|\ .[1 + sh + (sh)^2 + (sh)^3 + \cdots]$

$$= \frac{sh^{2n}}{1 - sh}\|x_0 - x_1\|$$

Now using the lemma 2.13 and taking limit $n \to \infty$ we get

$$\lim_{n\to\infty}\|x_{2n} - x_{2m}\| = 0$$

$\{x_{2n}\}$ is a Cauchy sequence in X.

Since X is complete we consider that $\{x_{2n}\}$convergence to x^* .Now we show that x^* is fixed point of T_1.

$$\|x^* - T_1x^*\| \le s[\|x^* - x_{2n}\| + \|x_{2n} - T_1x^*\|]$$

$\le s[\|x^* - x_{2n}\| + \| T_1x_{2n-1} - T_1x^*\|\ ]$

$$\le s[\|x^* - x_{2n}\| + a\|x^* - T_1x^*\| + b\|x_{2n-1} - T_1x_{2n-1}\| + c\|x_{2n-1} - x^*\|]$$

$$(1 - as)\|x^* - T_1x^*\| \le s[\|x^* - x_{2n}\| + b\|x_{2n-1} - T_1x_{2n-1}\| + c\|x_{2n-1} - x^*\|]$$

$$\|x^* - T_1x^*\| \le \frac{s}{(1 - as)}[\|x^* - x_{2n}\| + bh^{2n}\|x_0 - T_1x_1\| + c\|x_{2n-1} - x^*\|]$$

Taking limit $n \to \infty$we get,

$$\lim_{n\to\infty}\|x^* - T_1x^*\| = 0$$

$$x^* = T_1x^*$$

Hence x^* is a fixed point of T_1.

Now,if z be another fixed point of T_1.

T_1z = z.Then

$\|x^* - T_1z\| \le s[\|x^* - x_{2n}\| + \|x_{2n} - T_1z\|]$

$\le s[\|x^* - x_{2n}\| + \| T_1x_{2n-1} - T_1z\|\ ]$

$$\le s[\|x^* - x_{2n}\| + a\|x^* - T_1z\| + b\|x_{2n-1} - T_1x_{2n-1}\| + c\|x_{2n-1} - x^*\|]$$

$$(1 - as)\|x^* - T_1x^*\| \le s[\|x^* - x_{2n}\| + b\|x_{2n-1} - T_1x_{2n-1}\| + c\|x_{2n-1} - x^*\|]$$

$$\|x^* - T_1x^*\| \le \frac{s}{(1 - as)}[\|x^* - x_{2n}\| + bh^{2n}\|x_0 - T_1x_1\| + c\|x_{2n-1} - x^*\|]$$

Taking limit $n \to \infty$we get,

$$\lim_{n\to\infty}\|x^* - T_1z\| = 0$$

$x^* = T_1z$ = z

$x^* = z$

Therefore T_1 has a unique fixed point.

Similarly it can be establishd that T_2x^*=x^*.

Hence T_1x^* = x^* =T_2x^*

Thus x^* is the unique common fixed point of T_1 and T_2.

These completed the proof of the theorem.

Theorem3.2: Let X be a Generalized Banach space with $\|.,.\|$ and let $T_1, T_2: X \to X$ be a function with the following mapping:

$\|T_1(x) - T_2(y)\| \le a_1\|x - T_1x\| + a_2[\|y - T_2(y)\| + \|x - y\|]$ (3.2.1)

$\forall x, y \in X$Where a_1 & a_2 are non negative real number and satisfy $a + 2sb \leq 1 for s \geq 1$,then T_1 & T_2 have a unique common fixed point.

Proof: Let $x_0 \in X$ and $\{x_{2n}\}$ and $\{x_{2n+1}\}$ be any two sequence in X such that

$$x_{2n} = T_1 x_{2n-1} = {T_1}^{2n} x_0 \tag{3.2.2}$$

$$x_{2n+1} = T_2 x_{2n} = {T_2}^{2n+1} x_0 \tag{3.2.3}$$

Note that,if x_{2n}=x_{2n+1} for some n $\geq$ 0,then x_{2n} is fixed point of T_1 and T_2.Now putting $x = x_{2n}$ and $y = x_{2n-1}$.from(3.2.1),we have

$\|x_{2n+1} - x_{2n}\|$=$\|T_1 x_{2\,n} - T_2 x_{2n-1}\|$

$$\leq a_1\|x_{2n} - T_1 x_{2n}\| + a_2[\|x_{2n-1} - T_1 x_{2n-1}\| + \|x_{2n} - x_{2n-1}\|]$$

$$= a_1\|x_{2n} - x_{2n+1}\| + a_2[\|x_{2n-1} - x_{2n}\| + \|x_{2n} - x_{2n-1}\|]$$

$\Rightarrow \quad (1 - a_1)\|x_{2n} - x_{2n-1}\| \leq 2a_2\|x_{2n} - x_{2n-1}\|$

$$\Longrightarrow \|x_{2n} - x_{2n-1}\| \leq \left(\frac{2a_2}{1 - a_1}\right)\|x_{2n} - x_{2n-1}\|$$

$=h\|x_{2n} - x_{2n-1}\|$ where h=$\frac{a_2}{1-a_1}$

Continuing this process we can easily say that $\|x_{2n} - x_{2n-1}\| \leq h^{2n}\|x_1 - x_0\|$

This implies that T_1 and T_2 are contraction mapping. Now it is to show that $\{x_{2n}\}$ is Cauchy sequence in X.

Let m,n> 0 with $m > n$ then from (3.1.1)we have

$$\|x_{2n} - x_{2m}\| \leq s\{\|x_{2n} - x_{2n+1}\| + \|x_{2n+1} - x_{2m}\|\}$$

$\leq s\|x_{2n} - x_{2n+1}\| + s^2\left\|x_{2n+1} - x_{2n+|2}\right\| + s^3 \left\|x_{2n+|2} - x_{2n+3}\right\| + \ldots\ldots\ldots\ldots\ldots\ldots.$

$\leq sh^{2n}\|x_0 - x_1\| + s^2h^{2n+1}\|x_0 - x_1\| + s^3h^{2n+2}\|x_0 - x_1\| + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= sh^{2n}\,\|x_0 - x_1\|\ .[1 + sh + (sh)^2 + (sh)^3 + \cdots]$

$$= \frac{sh^{2n}}{1 - sh}\|x_0 - x_1\|$$

Now using the lemma 2.13 and taking limit $n \to \infty$ we get

$$\lim_{n\to\infty}\|x_{2n} - x_{2m}\| = 0$$

$\{x_{2n}\}$ is a Cauchy sequence in X.

Since X is complete we consider that $\{x_{2n}\}$convergence to x^* .Now we show that x^* is fixed point of T_1.

$$\|x^* - T_1x^*\| \leq s[\|x^* - x_{2n}\| + \|x_{2n} - T_1x^*\|]$$

$\leq s[\|x^* - x_{2n}\| + \|\,T_1x_{2n-1} - T_1x^*\|\ ]$

$$\leq s[\|x^* - x_{2n}\| + a_1\|x^* - T_1x^*\| + a_2[\|x_{2n-1} - T_1x_{2n-1}\| + \|x_{2n-1} - x^*\|]]$$

$$(1 - a_1s)\|x^* - T_1x^*\| \leq s[\|x^* - x_{2n}\| + a_2[\|x_{2n-1} - T_1x_{2n-1}\| + \|x_{2n-1} - x^*\|]]$$

$$\|x^* - T_1x^*\| \leq \frac{s}{(1 - a_1s)}[\|x^* - x_{2n}\| + a_2h^{2n}\|x_0 - T_1x_1\| + \|x_{2n-1} - x^*\|]$$

Taking limit $n \to \infty$we get,

$$\lim_{n\to\infty}\|x^* - T_1x^*\| = 0$$

$$x^* = T_1x^*$$

Hence x^* is a fixed point of T_1.

Now,if z be another fixed point of T_1.

T_1z = z.Then

$\|x^* - T_1 z\| \le s[\|x^* - x_{2n}\| + \|x_{2n} - T_1 z\|]$

$\le s[\|x^* - x_{2n}\| + \| T_1 x_{2n-1} - T_1 z\|]$

$$\le s[\|x^* - x_{2n}\| + a_1\|x^* - T_1 z\| + a_2[\|x_{2n-1} - T_1 x_{2n-1}\| + \|x_{2n-1} - x^*\|]]$$

$$(1 - a_1 s)\|x^* - T_1 x^*\| \le s[\|x^* - x_{2n}\| + a_2[\|x_{2n-1} - T_1 x_{2n-1}\| + \|x_{2n-1} - x^*\|]]$$

$$\|x^* - T_1 x^*\| \le \frac{s}{(1 - a_1 s)}[\|x^* - x_{2n}\| + a_2[h^{2n}\|x_0 - T_1 x_1\| + \|x_{2n-1} - x^*\|]]$$

Taking limit $n \to \infty$we get,

$$\lim_{n\to\infty}\|x^* - T_1 z\| = 0$$

$x^* = T_1 z$ = z

$x^* = z$

Therefore T_1 has a unique fixed point.

Similarly it can be establishd that $T_2 x^*$=x^*.

Hence $T_1 x^*$ = x^* =$T_2 x^*$

Thus x^* is the unique common fixed point of T_1 and T_2.

These completed the proof of the theorem.

Theorem 3.3: Let X be a generalized Banach space with $\|.,.\|$ and $T_1, T_2 : X \to X$ be a function satisfied the following condition for all x,y in X,

$\|T_1(x) - T_2(y)\| \le a\|x - T_1 x\| + b\|y - T_2 y\| + c\|x - T_1 y\| + e\|y - T_2 y\| + f\|x - y\|$

Where a,b,c,e and f are non negative real number & satisfy $\alpha = a + b + c + e + f$ such that $\alpha \in \left(0, \frac{1}{2s}\right)$ for s≥ 1,then T_1 & T_2 have a unique common fixed point.before going to prove this theorem we required following lemma 3.4

Lemma 3.4: Let he condition (3.3.1) hold on generalized Banach space for self map T_1 *and* T_2on it.Then if $\alpha \in \left(0, \frac{1}{2s}\right)$

there exits $\beta < \frac{1}{2s}$ such that $\left\|T_1 x - {T_1}^2 x\right\| \le \|x - T_1 x\|$ (3.4.1)

and $\left\| T_2 x - {T_2}^2 x\right\| \le \|x - T_2 x\|$ (3.4.2)

Proof of the theorem 3.3

Let $x_0 \in X$ and $\{x_{2n}\}$be a sequence in X such that

$x_{2n} = T_1 x_{2n-1} = {T_1}^{2n} x_0$

x_{2n+1} =$T_2 x_{2n}$=${T_2}^{2n+1} x_1$

Now using lemma 3.4 we can show that

$$\|x_{2n+1} - x_{2n}\| \le \beta^{2n}\|x_0 - x_1\|$$

Now we show that $\{x_{2n}\}$ is a Cauchy sequence in X.Let $m, n > 0$ with $m > n$

$$\|x_{2n} - x_{2m}\| \le s[\![\|x_{2n} - x_{2n+1}\| + \|x_{2n+1} - x_{2m}\|]\!]$$

$\le s\|x_{2n} - x_{2n+1}\| + s^2\|x_{2n+1} - x_{2n+2}\| + s^3\|x_{2n+2} - x_{2n+3}\| + \ldots\ldots\ldots$

$\le s\beta^{2n}\|x_0 - x_1\| + s^2\beta^{2n+1}\|x_0 - x_1\| + s^3\beta^{2n+2}\|x_0 - x_1\| + \ldots\ldots$

When taking limit $n \to \infty$ we get

$$\lim_{n\to\infty} \|x_{2n} - x_{2m}\| = 0$$

$\{x_{2n}\}$ is Cauchy sequence in X.

Since X is complete we consider that $\{x_{2n}\}$ converges to x^* .Now we show that x^* is fixed point of T_1.

$$\|x^* - T_1x^*\| \le s[\|x^* - x_{2n}\| + \|x_{2n} - T_1\, x^*\|]$$

$$\le s[\|x^* - x_{2n}\| + \|T_1\, x_{2n-1} - T_1\, x^*\|]$$

$$\le s[\|x^* - x_{2n}\| + a\|x_{2n-1} - T_1\, x_{2n-1}\| + b\|x^* - T_1\, x^*\| + c\|x_{2n-1} - T_1\, x^*\| + e\|x^* - T_1\, x_{2n-1}\| + f\|x_{2n-1} - x^*\|]$$

$$\|x^* - T_1x^*\| \le s[a\|x_{2n-1} - x_{2n}\| + b\|x^* - T_1\, x^*\| + c\|x_{2n-1} - T_1\, x^*\| + (e+1)\|x^* - x_{2n}\| + f\|x_{2n-1} - x^*\|]$$

Taking limit $n \to \infty$ we get

$$\|x^* - T_1\, x^*\| \le s(b+c)\|x^* - T_1\, x^*\|$$

Which is contradiction unless $x^* = T_1x^*$

Hence x^* is a fixed point of T_1.

Now if z be another fixed point of T_1.

T_1z = z.Then

$$\|x^* - T_1z\| \le s[\|x^* - x_{2n}\| + \|x_{2n} - T_1z\|]$$

$$\le s[\|x^* - x_{2n}\| + \| T_1x_{2n-1} - T_1z\|$$

$$\le s[\|x^* - x_{2n}\| + a\|x^* - T_1z\| + b\|x_{2n-1} - T_1x_{2n-1}\| + c\|x^* - T_1x_{2n-1}\| + e\|x_{2n-1} - T_1x^*\| + f\|x_{2n-1} - x^*\|]$$

$$\Rightarrow (1-as)\|x^* - T_1z\| \le s[\|x^* - x_{2n}\| + b\|x_{2n-1} - T_1x_{2n-1}\| + c\|x^* - T_1x_{2n-1}\| + e\|x_{2n-1} - T_1x^*\| + f\|x_{2n-1} - x^*\|$$

$$\|x^* - T_1z\| \le \frac{s}{(1-as)}\|x^* - x_{2n}\| + b\|x_{2n-1} - T_1x_{2n-1}\| + c\|x^* - T_1x_{2n-1}\| + e\|x_{2n-1} - T_1x^*\| + f\|x_{2n-1} - x^*\|$$

Taking limit $n \to \infty$we get,

$$\lim_{n\to\infty} \|x^* - T_1z\| = 0$$

$x^* = T_1z$ = z

$x^* = z$

Therefore T_1 has a unique fixed point.

Similarly it can be establishd that T_2x^*=x^*.

Hence T_1x^* = x^* =T_2x^*

Thus x^* is the unique common fixed point of T_1 and T_2.

These completed the proof of the theorem.

Thorem3.5: Let X be a generalized Banach space with $\|.,.\|$ and T_1, T_2:$X \to X$ be a function satisfied the following condition for all x,y in X,

$$\|T_1\,(x) - T_2(y)\| \le a_1\|x - T_1\, x\| + a_2[\|y - T_2y\| + \|x - T_1\, y\|] + a_3[\|y - T_2y\| + \|x - y\|]$$

Where a_1, a_2 *and* a_3 are non negative real number & satisfy $\alpha = a_1 + a_2 + a_3$ such that $\alpha \in \left(0, \frac{1}{2s}\right)$ for s$\geq$ 1,then T_1 & T_2 have a unique common fixed point.

Proof: Let $x_0 \in X$ and $\{x_{2n}\}$be a sequence in X such that

$x_{2n} = T_1x_{2n-1} = {T_1}^{2n}x_0$

x_{2n+1} $=T_2 x_{2n}$$=T_2{}^{2n+1}x_1$

Now using lemma 3.4 we can show that

$$\|x_{2n+1} - x_{2n}\| \le \beta^{2n}\|x_0 - x_1\|$$

Now we show that $\{x_{2n}\}$ is a Cauchy sequence in X.Let $m,n > 0$ with $m > n$

$$\|x_{2n} - x_{2m}\| \le s[\![\|x_{2n} - x_{2n+1}\| + \|x_{2n+1} - x_{2m}\|]\!]$$

$\le s\|x_{2n} - x_{2n+1}\| + s^2\|x_{2n+1} - x_{2n+2}\| + s^3\|x_{2n+2} - x_{2n+3}\| + \ldots\ldots\ldots$

$\le s\beta^{2n}\|x_0 - x_1\| + s^2\beta^{2n+1}\|x_0 - x_1\| + s^3\beta^{2n+2}\|x_0 - x_1\| + \ldots\ldots$

When taking limit $n \to \infty$ we get

$$\lim_{n\to\infty}\|x_{2n} - x_{2m}\| = 0$$

$\{x_{2n}\}$ is Cauchy sequence in X.

Since X is complete we consider that $\{x_{2n}\}$ converges to x^* .Now we show that x^* is fixed point of T_1.

$$\|x^* - T_1x^*\| \le s[\|x^* - x_{2n}\| + \|x_{2n} - T_1\, x^*\|]$$

$$\le s[\|x^* - x_{2n}\| + \|T_1\, x_{2n-1} - T_1\, x^*\|]$$

$$\le s\big[\|x^* - x_{2n}\| + a_1\|x_{2n-1} - T_1\, x_{2n-1}\| + a_2[\|x^* - T_1\, x^*\| + \|x_{2n-1} - T_1\, x^*\|] + a_3[\|x^* - T_1\, x_{2n-1}\| + \|x_{2n-1} - x^*\|]\big]$$

$$\|x^* - T_1x^*\| \le s[\|x^* - x_{2n}\| + a_1\|x_{2n-1} - T_1\, x_{2n-1}\| + a_2[\|x^* - T_1\, x^*\| + \|x_{2n-1} - T_1\, x^*\|] + (a_3 + 1)\|x^* - x_{2n}\| + a_3\|x_{2n-1} - x^*\|]$$

Taking limit n $\to \infty$ we get

$\|x^* - T_1x^*\| \le sa_2\|x^* - T_1x^*\|$

Which is contradiction unless $x^* = T_1x^*$

Hence x^* is a fixed point of T_1.

Now if z be another fixed point of T_1.

T_1z = z.Then

$\|x^* - T_1z\| \le s[\|x^* - x_{2n}\| + \|x_{2n} - T_1\, z\|]$

$\le$s$[\|x^* - x_{2n}\| + \|\, T_1x_{2n-1} - \, T_1z\|$

$\le s[\|x^* - x_{2n}\| + a_1\|x^* - T_1z\|$$+a_2[\|x_{2n-1} - T_1x_{2n-1}\| + \|x^* - T_1x_{2n-1}\|]$$+a_3[\|x_{2n-1} - T_1x^*\| + \|x_{2n-1} - x^*\|]]$

$\Longrightarrow (1 - a_1s)\|x^* - T_1z\| \le s[\|x^* - x_{2n}\| + a_2[\|x_{2n-1} - T_1x_{2n-1}\| + \|x^* - T_1x_{2n-1}\|] + a_3[\|x_{2n-1} - T_1x^*\| + \|x_{2n-1} - x^*\|]$

$$\|x^* - T_1z\| \le \frac{s}{(1 - a_1s)}[\|x^* - x_{2n}\| + a_2[\|x_{2n-1} - T_1x_{2n-1}\| + \|x^* - T_1x_{2n-1}\|] + a_3[\|x_{2n-1} - T_1x^*\| + \|x_{2n-1} - x^*\|]]$$

Taking limit $n \to \infty$we get,

$$\lim_{n\to\infty}\|x^* - T_1z\| = 0$$

$x^* = T_1z$ = z

$x^* = z$

Therefore T_1 has a unique fixed point.

Similarly it can be establishd that T_2x^*=x^*.

Hence T_1x^* $= x^*$ $=T_2x^*$

Thus x^* is the unique common fixed point of T_1 and T_2.

These completed the proof of the theorem.

Theorem 3.6: Let X be a generalized Banach space with $\|.,.\|$ and let $T_{1,}T_2: X \to X$ be a function with the following mapping:

$\|T_1\,(x) - T_2(y), z\| \le a\|x - T_1\,x, z\| + b\|y - T_2 y, z\| + c\|x - y, z\|$ (3.6.1)

$\forall x, y, z \in M$, where a,b and c are non negative real number &satisfy a+s(b+c)< 1 for $s \ge 1$,then T_1 *and* T_2 have a unique common fixed point.

Proof: Let $x_0 \in M$ and $\{x_{2n}\}$ be a sequence in M such that $x_{2n} = T_1 x_{2n-1} = {T_1}^{2n} x_0$

$$x_{2n+1}=T_2 x_{2n} = {T_2}^{2n+1} x_1$$

$$\|x_{2n+1,} - x_{2n}, z\| = \|T_1\,x_{2n} - T_1\,x_{2n-1}, z\|$$

$\le a\|x_{2n} - T_1\,x_{2n}, z\| + b\ \|x_{2n-1} - T_1 x_{2n-1}, z\| + c\|x_{2n} - x_{2n-1}, z\|$

$$= a\|x_{2n} - x_{2n+1,}, z\| + b\|x_{2n-1} - T_1 x_{2n-1}, z\| + c\|x_{2n} - x_{2n-1}, z\|$$

$$\Rightarrow (1-a)\|x_{2n} - x_{2n+1,}, z\| \le (b + c)\|x_{2n} - x_{2n-1}, z\|$$

$$\Rightarrow \|x_{2n} - x_{2n+1,} z\| \le \left(\frac{b+c}{1-a}\right)\|x_{2n} - x_{2n-1}, z\|$$

$$= h\|x_{2n} - x_{2n-1}, z\|$$

Continuing this process we can easily say that $\|x_{2n} - x_{2n+1}, z\| \le h^{2n}\|x_{2n} - x_{2n-1}, z\|$ $\le$ $h^{2n}\|x_1 - x_0, z\|$

This implies that T_1 *and* T_2 are contraction mapping.Now it is to show that $\{x_{2n}\}$ is Cauchy sequence in X.

Let $m, n > 0$ with $m > n$ then from (3.5.1) we have ,

$$\|x_{2n} - x_{2m}, z\| \le s\{\|x_{2n} - x_{2n+1}, z\| + \|x_{2n+1} - x_{2m}, z\|\}$$

$$\le s\|x_{2n} - x_{2n+1}, z\| + s^2\|x_{2n+1} - x_{2n+2}, z\| + s^3\|x_{2n+2} - x_{2n+3}, z\|$$

$$\le sh^{2n}\|x_0 - x_1, z\| + s^2 h^{2n+1}\|x_0 - x_1, z\| + s^3 h^{2n+2}\|x_0 - x_1, z\|$$

$$= sh^{2n}\|x_0 - x_1, z\|[1 + sh + (sh)^2 + (sh)^3 + \cdots]$$

$$= \frac{sh^{2n}}{1 - sh}\|x_0 - x_1, z\|$$

Now using the lemma and taking limit $n \to \infty$ we get $\lim_{n\to\infty}\|x_{2n} - x_{2m}, z\| = 0$

$\{x_{2n}\}$ is a Cauchy sequence in X.

Since X is complete we concider that $\{x_{2n}\}$ converges to x^*.Now we show that x^* is fixed point of T_1.

$$\|x^* - T_1\,x^*, z\| \le s[\|x^* - x_{2n}, z\| + \|x_{2n} - T_1 x^*, z\|]$$

$$\le s[\|x^* - x_{2n}, z\| + \|T_1 x_{2n-1} - T_1\,x^*, z\|]$$

$$\le s[\|x^* - x_{2n}, z\| + a\|x^* - T_1 x^*, z\| + b\|x_{2n-1} - T_1 x_{2n-1}, z\| + c\|x_{2n-1} - x^*, z\|]$$

$$\Rightarrow (1\text{-}as)\|x^* - T_1 x^*, z\| \le s[\|x^* - x_{2n}, z\| + b\|x_{2n-1} - T_1 x_{2n-1}, z\| + c\|x_{2n-1} - x^*, z\|]$$

$$\|x^* - T_1 x^*, z\| \le \frac{s}{(1 - as)}[\|x^* - x_{2n}, z\| + b\|x_{2n-1} - T_1 x_{2n-1}, z\| + c\|x_{2n-1} - x^*, z\|]$$

Taking limit $n \to \infty$ we get,

$$\lim_{n\to\infty}\|x^* - T_1 x^*, z\| = 0$$

$$x^* = T_1 x^*$$

x^* is fixed point of T_1.

Now,if z be another fixed point of T_1,

T_1z = z.then

$\|x^* - T_1 z, z\| \leq s[\|x^* - x_{2n}, z\| + \|x_{2n} - T_1 z, z\|]$

$\leq s[\|x^* - x_{2n}, z\| + \|T_1 x_{2n-1} - T_1 z, z\|]$

$\leq s[\|x^* - x_{2n}, z\| + a\|x^* - T_1 z, z\| + b\|x_{2n-1} - T_1 x_{2n-1}, z\| + c\|x_{2n-1} - x^*, z\|]$

$\Rightarrow$(1-as)$\|x^* - T_1 z, z\| \leq s[\|x^* - x_{2n}, z\| + b\|x_{2n-1} - T_1 x_{2n-1}, z\| + c\|x_{2n-1} - x^*, z\|]$

$\Rightarrow \|x^* - T_1 z, z\| \leq \frac{s}{(1-as)}[\|x^* - x_{2n}, z\| + b\|x_{2n-1} - T_1 x_{2n-1}, z\| + c\|x_{2n-1} - x^*, z\|]$

Taking limit n$\longrightarrow \infty$ we get

$$\lim_{n\to\infty} \|x^* - T_1 z, z\| = 0$$

$$\Longrightarrow x^* = T_1 z = z$$

$$\Rightarrow \quad x^* = z$$

Therefore T_1 has a unique fixed point.

Similarly it can be establishd that $T_2 x^*$=x^*.

Hence $T_1 x^*$ = x^* =$T_2 x^*$

Thus x^* is the unique common fixed point of T_1 and T_2.

These completed the proof of the theorem.

Theorem3.7: Let X be a generalized Banach space with $\|.,.\|$ and let $T_{1,}T_2: X \to X$ be a function with the following mapping:

$\|T_1(x) - T_2(y), z\| \leq a_1\|x - T_1 x, z\| + a_2[\|y - T_2 y, z\| + \|x - y, z\|]$ (3.7.1)

$\forall x, y, z \in M$,where a_1 *and* a_2 are non negative real number &satisfy for $s \geq 1$,then T_1 *and* T_2 have a unique common fixed point.

Proof: Let $x_0 \in M$ and $\{x_{2n}\}$ be a sequence in M such that $x_{2n} = T_1 x_{2n-1} = {T_1}^{2n} x_0$

x_{2n+1}=$T_2 x_{2n}$ = ${T_2}^{2n+1} x_1$

$$\|x_{2n+1,} - x_{2n}, z\| = \|T_1 x_{2n} - T_1 x_{2n-1}, z\|$$

$\leq a_1\|x_{2n} - T_1 x_{2n}, z\| + a_2[\|x_{2n-1} - T_1 x_{2n-1}, z\| + \|x_{2n} - x_{2n-1}, z\|]$

$= a_1\|x_{2n} - x_{2n+1,}, z\| + a_2[\|x_{2n-1} - x_{2n}, z\| + \|x_{2n} - x_{2n-1}, z\|]$

$\Rightarrow$(1-a_1)$\|x_{2n} - x_{2n+1,}, z\| \leq a_2\|x_{2n} - x_{2n-1}, z\|$

$$\Rightarrow \|x_{2n} - x_{2n+1,} z\| \leq \left(\frac{a_2}{1 - a_1}\right)\|x_{2n} - x_{2n-1}, z\|$$

$$= h\|x_{2n} - x_{2n-1}, z\|$$

Continuing this process we can easily say that $\|x_{2n} - x_{2n+1}, z\| \leq h^{2n}\|x_{2n} - x_{2n-1}, z\|$

$\leq$

$h^{2n}\|x_1 - x_0, z\|$

This implies that T_1 *and* T_2 are contraction mapping. Now it is to show that $\{x_{2n}\}$ is Cauchy sequence in X.

Let $m, n > 0$ with $m > n$ then from (3.5.1) we have ,

$$\|x_{2n} - x_{2m}, z\| \leq s\{\|x_{2n} - x_{2n+1}, z\| + \|x_{2n+1} - x_{2m}, z\|\}$$

$$\leq s\|x_{2n}-x_{2n+1},z\| + s^2\|x_{2n+1}-x_{2n+2},z\| + s^3\|x_{2n+2}-x_{2n+3},z\|$$

$$\leq sh^{2n}\|x_0-x_1,z\| + s^2h^{2n+1}\|x_0-x_1,z\| + s^3h^{2n+2}\|x_0-x_1,z\|$$

$$= sh^{2n}\|x_0-x_1,z\|[1+sh+(sh)^2+(sh)^3+\cdots]$$

$$= \frac{sh^{2n}}{1-sh}\|x_0-x_1,z\|$$

Now using the lemma and taking limit $n\to\infty$ we get $\lim_{n\to\infty}\|x_{2n}-x_{2m},z\| = 0$

$\{x_{2n}\}$ is a Cauchy sequence in X.

Since X is complete we concider that $\{x_{2n}\}$ converges to x^*.Now we show that x^* is fixed point of T_1.

$$\|x^*-T_1x^*,z\| \leq s[\|x^*-x_{2n},z\| + \|x_{2n}-T_1x^*,z\|]$$

$$\leq s[\|x^*-x_{2n},z\| + \|T_1x_{2n-1}-T_1x^*,z\|]$$

$$\leq s\big[\|x^*-x_{2n},z\| + a_1\|x^*-T_1x^*,z\| + a_2[\|x_{n-1}-T_1x_{2n-1},z\| + \|x_{2n-1}-x^*,z\|]\big]$$

$\Longrightarrow$(1-a_1s)$\|x^*-T_1x^*,z\| \leq s\big[\|x^*-x_{2n},z\| + a_2[\|x_{2n-1}-T_1x_{2n-1},z\| + \|x_{2n-1}-x^*,z\|]\big]$

$$\|x^*-T_1x^*,z\| \leq \frac{s}{(1-a_1s)}\big[\|x^*-x_{2n},z\| + a_2[h^{2n}\|x_0-T_1x_1,z\| + \|x_{2n-1}-x^*,z\|]\big]$$

Taking limit $n\to\infty$ we get,

$$\lim_{n\to\infty}\|x^*-T_1x^*,z\| = 0$$

$$x^* = T_1x^*$$

x^* is fixed point of T_1.

Now,if z be another fixed point of T_1,

T_1z = z.then

$\|x^*-T_1z,z\| \leq s[\|x^*-x_{2n},z\| + \|x_{2n}-T_1z,z\|]$

$$\leq s[\|x^*-x_{2n},z\| + \|T_1x_{2n-1}-T_1z,z\|]$$

$$\leq s\big[\|x^*-x_{2n},z\| + a_1\|x^*-T_1z,z\| + a_2[\|x_{2n-1}-T_1x_{2n-1},z\| + \|x_{2n-1}-x^*,z\|]\big]$$

$\Longrightarrow$(1-a_1s)$\|x^*-T_1z,z\| \leq s\big[\|x^*-x_{2n},z\| + a_2[\|x_{2n-1}-T_1x_{2n-1},z\| + \|x_{2n-1}-x^*,z\|]\big]$

$\Rightarrow \|x^*-T_1z,z\| \leq \frac{s}{(1-a_1s)}\big[\|x^*-x_{2n},z\| + a_2h^{2n}[\|x_0-T_1x_1,z\| + \|x_{2n-1}-x^*,z\|]\big]$

Taking limit n$\longrightarrow\infty$ we get

$$\lim_{n\to\infty}\|x^*-T_1z,z\| = 0$$

$$\Longrightarrow x^* = T_1z = z$$

$$\Rightarrow x^* = z$$

Therefore T_1 has a unique fixed point.

Similarly it can be establishd that T_2x^*=x^*.

Hence T_1x^* = x^* =T_2x^*

Thus x^* is the unique common fixed point of T_1 and T_2.

These completed the proof of the theorem.

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